# PULSATION SPECTRUM GENERATION DURING SUBHARMONIC TRANSITION IN BOUNDARY LAYERS

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As shown in [1-5], an s-type laminar-turbulent transition universally takes place in a boundary layer for initially small, intense disturbances in the layer. The salient features of this regime are explained within the framework of nonlinear stability theory [6]. The mechanism of resonant interaction of Tollmien–Schlichting waves (TS) plays a leading role, which in the initial stages leads to selection of a pair of three-dimensional waves from the background pulsations [7]. The parameters of the latter [frequency $\omega$ and wave vectors $(\alpha, \pm\beta)$] correspond to the maximum rate of parametric amplification in the field of the introduced two-dimensional TS disturbance with $\omega_0 = 2\omega$ and $(\alpha_0, 0)$. The selected symmetric triad forms the fundamental structure of the s-transition. When the intensities of the triad components are equalized, the parametric stage is transformed to a nonlinear stage, in which there is explosive amplification of all interacting waves. According to experiments [2, 4, 5], approach to the explosive regime is accompanied by rapid broadening of the low-frequency (LF) part of the spectrum of spatial pulsations. Below we note the stabilization of the amplitude level and transition to turbulent motion.

We know from the theory of nonlinear systems that there is a close relation between the processes leading to stochastic behavior and the mechanism of spectrum filling during bounded growth of oscillations. It has been proposed that such processes can be realized in the boundary layer as a consequence of a resonant cascade transfer of energy along the spectrum in the region of strongly dissipating, "essentially" three-dimensional LF waves.

In this work, we study the possibility and consequences of resonant cascade transformation of the spectrum. The mechanism of spectrum filling is analyzed in comparison with experiments.

We examine a cascade process of excitation of background pulsations $\omega_n = \omega_0/2^n$ ($n = 1, 2, \ldots$) in a field of a given frequency $\omega_0$. The model includes a system of waves: induced plane waves with parameters $(\omega_0, \alpha_0, 0)$ of amplitude $A_0$, a pair of symmetric subharmonic waves $(\omega_0/2, \alpha_1, \pm\beta_1)$, $A_1$, and two pairs of secondary subharmonics $(\omega_0/4, \alpha_2, \pm\beta_2)$, $A_2$ and $(\omega_0/4, \alpha_3, \pm\beta_3)$, $A_3$. The perturbation of the velocity field of the flow $\varepsilon u = \varepsilon(u_1, u_2, u_3)$ can be represented as

$$u(x, y, z, t) = \sum_{j=0}^{m} B_j u_j \exp i\theta_j(x, z, t) + \varepsilon\Psi(x, y, z, t),$$

where $\theta_j = -\omega_j t + \beta_j z + \int \alpha_j\,dx$, $u_j(y)\left(\max\limits_{0\leqslant y<\infty}|u_j| = 1\right)$ and the dispersion relation $\omega_j + i\gamma_j = \Omega(\alpha_j, \beta_j)$ are determined by the local-parallel Orr–Sommerfeld problem [8]; $\Psi$ is a function quasi-periodic in $(x, z, t)$; and the parameter $\varepsilon \ll 1$. Under conditions of steady-state and transverse uniformity, the system of equations for complex amplitude $A_j = B_j e^{\gamma_j t}$ takes the form

$$(v_0 d/dx - \gamma_0) A_0 = S A_1^2 e^{-i\int\Delta_0 dx},$$
$$(v_1 d/dx - \gamma_1) A_1 = S_0 A_0 A_1^* e^{i\int\Delta_0 dx} + C_1 a_2^2 e^{-i\int\Delta_1 dx} + C_2 a_3^2 e^{-i\int\Delta_2 dx} + C_3 a_2 a_3 e^{-i\int\Delta_3 dx}, \quad (1)$$
$$(v_{2,3} d/dx - \gamma_{2,3} - i w_{2,3}\delta_{2,3}) a_{2,3} = S_{2,3} A_1 a_{2,3}^* e^{i\int\Delta_{1,2} dx} + D_{2,3} A_1 a_{3,2}^* e^{i\int\Delta_3 dx}.$$

Here $\delta_{2,3} = \beta_1/2 - \beta_{2,3}$; $\Delta_0 = \alpha_0 - 2\alpha_1$; $\Delta_1 = \alpha_1 - 2\alpha_2$; $\Delta_2 = \alpha_1 - 2\alpha_3$; $\Delta_3 = \alpha_1 - \alpha_2 - \alpha_3$; and

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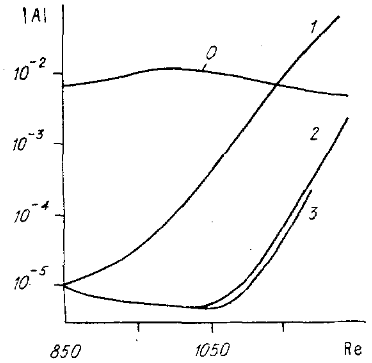


Fig. 1

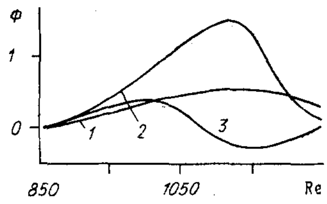


Fig. 2

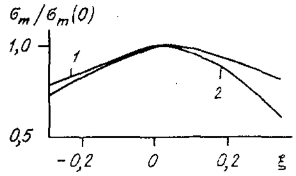


Fig. 3

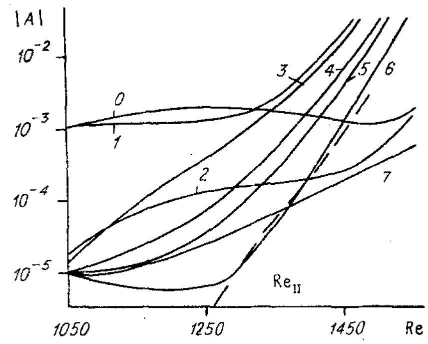


Fig. 4

the following substitution has been made $A_{2,3} = a_{2,3}e^{i\delta_{2,3}z}$. The coefficients v, w, S, C, and D are constructed from the solution of homogeneous and inhomogeneous Orr–Sommerfeld equations for the Blasius profile [6].

We study the behavior of (1) for various $\beta_2$, $\beta_3$ during which $\beta_1$ is chosen from the region of maximum parametric increments ($\beta_1/\alpha_1 \approx 2$) [7]. Figure 1 shows the evolution of $|A_j(\mathrm{Re})|$ (here and below, the curve number corresponds to the index of the wave) for $F_0 = \omega_0\mathrm{Re} = 122\cdot10^{-6}$, $b_2 = (\beta_2/\mathrm{Re})\cdot10^3 = 0.217$, $b_3 = 0.254$. Re is determined according to the displacement thickness. We observed parametric growth in the first $\omega_0/2$ and second $\omega_0/4$ subharmonics. In this case, the second subharmonic lies outside synchronization with the two-dimensional TS wave frequency $\omega_0$, and is pumped by the three-dimensional wave $\omega_0/2$. In the Re range considered, the waves of frequencies $\omega_n$ ($n > 1$) lie in the zone of linear damping. The decrement grows with increasing n. This leads to the appearance of the threshold character of LF-wave pumping: the wave frequency $\omega_{n+1}$ is parametrically increased only after the amplitude of the n-th wave exceeds the threshold value. As shown by calculations, the ratio $(\beta/\alpha)_{2,3}$ grows with increasing n, which corresponds to the maximum parametric increment (curves 2 and 3 correspond to $\beta_2/\alpha_2 = 2.8$, $\beta_3/\alpha_3 = 3.44$ at the initial point).

Thus, a cascade process occurs, with sequential excitation of all "more than three-dimensional" subharmonics. The interaction is of an explosive nature, with synchronization of phases ($\varphi_j = \arg A_j$) $\Phi_1 = \varphi_0 - 2\varphi_1 + \int\Delta_1 dx$, $\Phi_2 = \varphi_1 - 2\varphi_2 + \int\Delta_2 dx$, $\Phi_3 = \varphi_1 - 2\varphi_3 + \int\Delta_3 dx$ (Fig. 2, curves 1-3). Growth stabilization in the cascade process does not take place.

The scenario for transition to turbulence in a boundary layer by period doubling is observed in the experiments of [5]. In that work, sequential excitation of three subharmonics $\omega_0/2$, $\omega_0/4$, $\omega_0/8$ is observed, with broadening and filling of the LF band of the spectrum and its subsequent transformation to a continuum. Broadening and filling of the LF spectrum can take place as a consequence of parametric interaction in asymmetric triplets, where $\omega_{k,l} \neq \omega_0/2$. A particularly effective interaction is preserved during synchronization of the frequencies $\omega_k + \omega_l = \omega_0$. Examples of interactions in such configurations, when they dominate the structure of the perturbation field are given in [9]. We will dwell on this in more detail.

We study the dependence of the increments of LF-perturbation growth $\sigma_{k,l} = \frac{1}{|A_{k,l}|}\frac{d|A_{k,l}|}{dx}$ on the frequency parameters $\omega_k$ and $\omega_l$. For convenience, we introduce the quantities $\pm\xi = \frac{\omega_0/2 - \omega_{k,l}}{\omega_0/2}$, which characterize the frequency wave selection from the subharmonic triplet. The model includes the fundamental wave ($\omega_0$, $\alpha_0$, 0) and three-dimensional pairs of waves ($\omega_1$, $\alpha_1$, $\pm\beta_1$), ($\omega_2$, $\alpha_2$, $\pm\beta_2$) of background intensity, where $\omega_{1,2} = (\omega_0/2)(1 - \xi)$. The system of amplitude equations in this case (m = 5), which takes into account cross coupling, is given in [9]. For fixed $\omega_0$ and $|\xi|$, the increments $\sigma_{1,2}$ depend on the initial amplitude of the pumping wave $|A_0(Re_0)|$ and the parameters $\beta_1$, $\beta_2$. Calculations of the increments $\sigma_{1,2}$ for various orientations of the wave vectors $(\beta/\alpha)_{1,2}$ show that in analogy with the subharmonic triplet [7], the increment $\sigma_{1,2}$ has a maximum at certain ($\beta_1^+$, $\beta_2^+$): $\sigma_m(|A_0|, \xi) = \sigma_{1,2}(\beta_1^+, \beta_2^+, |A_0|, \xi)$. The dependence $\sigma_m(\xi)/\sigma_m(0)$ is shown in Fig. 3 for $|A_0(Re_{II})| = 0.67$; 0.5%, and $F_0 = 115\cdot10^{-6}$ (curves 1 and 2, $Re_{II}$ is the Reynolds number on the upper branch of the neutral stability curve). For fixed $|A_0(Re_{II})|$, the maximum increment $\sigma_m(0)$ corresponds to symmetric subharmonics, and for $\xi \neq 0$, it slowly decreases with increasing wave selection $\xi$. A perturbation of lower frequency ($\xi > 0$) has a large increment. The width of the frequency band being effectively excited increases with growing intensity of the pumping wave $|A_0(Re_{II})|$. We can conclude that a large resonance width is capable of exciting resonant frequencies over a broad range of the spectrum.

These results make it possible to interpret the experimental data of [10], where waves were excited in a plate boundary layer. These consisted of a two-dimensional wave of frequency $\omega_0$ and a pair of symmetric three-dimensional waves of frequency $\omega_1 < \omega_0$ ($F_0 = 88\cdot10^{-6}$, $F_1 = 39.5\cdot10^{-6}$) with initial intensity on the order of 0.1%. A wide selection of LF spatial modes is observed downstream, whose intensity level reaches the induced levels. In addition to the induced waves $\omega_0$ and $\omega_1$, the waves of frequencies $2\omega_1$ and $\omega_0 - \omega_1$ also dominate in the initial stage. The transformation of the initially two-dimensional waves of frequency $2\omega_1$ to a three-dimensional wave in the region $Re \gtrsim Re_{II}$ is notable. It has been established that LF spatial modes are found in synchronization with the dominant modes $\omega_1$ and $2\omega_1$. The latter are the main conveyers of energy to the low frequencies, as asserted by [10].

We can explain the experimental results within the framework of the representation developed above. According to this, the introduction of the plane oscillation of frequency $\omega_0$ and the three-dimensional one of frequency $\omega_1 < \omega_0$ first of all leads to the selection of resonant spatial modes $\omega_7 = \omega_0/2$, $\omega_3 = \omega_0 - \omega_1$ from the background source oscillations. The plane mode $\omega_2 = 2\omega_1$ is also selected, as a consequence of the nonlinear interaction in the symmetric triplet $(\omega_1, \alpha_1, \beta_1) + (\omega_1, \alpha_1, -\beta_1) = (2\omega_1, \alpha_2, 0)$. This establishes the start of the process of cascading excitation of resonant frequencies $\omega_4 = \omega_2 - \omega_3$, $\omega_5 = \omega_0 - \omega_4$, $\omega_6 = \omega_2 - \omega_5$, and so on.

The results of calculating the amplitudes of the corresponding multi-wave system are shown in Fig. 4. Comparison with experiment [10] confirms the validity of the proposed model. In the initial stage $Re \lesssim 1250$, the modes with frequencies $\omega_0$, $\omega_1$, $\omega_2$, and $\omega_3$ dominate. The intensities $|A_2|$, $|A_3|$ of the waves being resonantly excited (of frequency $\omega_2$, $\omega_3$) in this region of the background are practically unrelated to their initial values ($|A_2(x_0)|$, $|A_3(x_0)|$) $\lesssim 10^{-4}$. In the region $Re \gtrsim 1350$, the intensities $|A_j|$ of the excited waves at frequencies $\omega_j$ (j = 4-6) reach the induced level and coincide with those obtained in the experiment. The intensity of the subharmonic (j = 7) is significantly less, which also agrees with the data from [10].

We emphasize that the experimental discovery of decreasing growth rate of the oscillations with increasing frequency separation $\omega_0 - \omega_1$ is in complete agreement with the idea that maximum interaction takes place in the subharmonic (symmetric) triads (see Fig. 3) [7]. Under conditions of pair interactions, the growth rate of beats is determined by the individual parameters of the coupled waves. Note that this model is concerned with resonant excitation of characteristic TS wave disturbances. In the case of resonance, the generation of oscillation $\omega_n$ (n-th cascade level) is not accompanied by an increase in the order of the interaction: $A(\omega_n) \sim \varepsilon A$. At the same time, for nonlinear harmonics (beats), the order must grow together with increasing n: $A(\omega_n) \sim (\varepsilon A)^n$ ($n \geq 2$), which is not observed in [10].

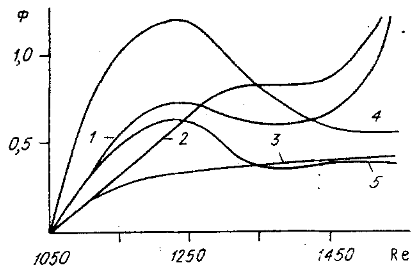


Fig. 5

The potentially competitive contribution of "nonresonant" oscillations in the perturbation spectrum can be introduced only through beats of the initial waves ($\omega_0$, $\omega_1$), described by the function $\Psi$ in the quadratic order of the theory. The behavior of the amplitude of noncharacteristic three-dimensional waves ($2\omega_1$, $2\alpha_1$, $\pm 2\beta$) is shown by the broken line in Fig. 4. Obviously, in the region Re $\gtrsim$ 1350, this three-dimensional wave exceeds the intensity of the plane wave of frequency $2\omega_1$ (curve 2), which explains the results of [10].

Calculations show that the two-dimensional wave of frequency $2\omega_1$ plays an important role in the process of energy transfer to low frequencies. In our model, all LF waves are parametrically coupled to the two plane waves $\omega_0$ and $\omega_1$. As is evident from the behavior of the phases in Fig. 5, the coupling with $\omega_2$ is decisive. The phases $\Phi_3 = \varphi_2 - 2\varphi_1$, $\Phi_4 = \varphi_2 - \varphi_3 - \varphi_4$, $\Phi_5 = \varphi_2 - \varphi_5 - \varphi_6$ (curves 3-5) demonstrate the nonlinear synchronization of LF perturbations with the phase $\varphi_2$ (phase localization). This is in contrast to the behavior of the phases $\Phi_1 = \varphi_0 - \varphi_3 - \varphi_6$, $\Phi_2 = \varphi_0 - \varphi_4 - \varphi_5$ (curves 1, 2), where such synchronization with $\varphi_0$ is not observed. The special role of the wave of frequency $\omega_2$ is emphasized in [10].

The character of the curves in Fig. 4 indicates that stabilization of amplitude growth does not take place in the process of energy transfer to low frequencies. The excitation of subsequent oscillations $\omega_n$ only weakly affects the behavior of waves of frequency $\omega_{n-1}$. This makes it possible to consider the behavior of the curves as invariant with respect to subsequent increase in the cascade level.

Our work shows that within the framework of weakly nonlinear theory, the process of spectrum filling, which precedes transition, can be explained. With growing intensity, there occurs both resonant excitation of low frequencies and generation of high-frequency spatial harmonics [11]. However, this process does not directly lead to stochastic motion, since the growth in intensity preserves the phase synchronization of the perturbations. The stabilization mechanism of the latter evidently is brought about primarily by the nonlinear generation of higher harmonics, and the transport and dissipation of energy in the high-frequency part of the spectrum.

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